Numerical Solution of Transonic Stream Function Equation

Mohamed Hafez* and Donald Lovell†
George Washington University, NASA Langley Research Center, Hampton, Virginia

The stream function equation, in conservation form, looks similar to the full potential equation and existing methods (e.g., artificial compressibility) can be readily applied. Rotational flows can be calculated once the vorticity (due to shocks or nonuniformity) is evaluated. There are, however, two main difficulties. The first is that the density is not uniquely determined in terms of the flux (there are two solutions: the subsonic and the supersonic branch with a square root singularity at the sonic point). Methods to overcome this difficulty are studied and results are presented with some remarks on inviscid separation and closed streamlines. The second difficulty, the need of two stream functions for three-dimensional calculations, is briefly discussed.

Introduction

NUMERICAL solution of Euler equations in the transonic regime is expensive when compared to a potential calculation mainly because of the existence of fast iterative algorithms for second-order potential equations. The potential assumption is not, however, valid for some cases of practical interest. It is required either to construct a fast relaxation method for the solution of steady Euler equations or to augment the potential calculations with the rotational effects in some way.

For two-dimensional and axisymmetric flows, Euler equations can be replaced by a second-order partial differential equation in terms of a stream function and the fact that the vorticity downstream of the shock is proportional to the pressure along a streamline. The stream function equation is of the mixed type and looks similar to the potential equation. The main difficulty with the stream function formulation is that the density is not uniquely determined in terms of the mass flux. The purpose of this study is to develop techniques to circumvent this obstacle and hence provide efficient algorithms for solving rotational flows using standard potential methods. It should be mentioned that in 1944 Emmons[1-3] solved the same equations by hand relaxation in which the shock was fitted.

Formulation of the Transonic Flow Problem in Terms of a Stream Function

Euler equations represent conservation of mass, momentum, and energy. The continuity equation for two-dimensional and axially symmetric flows, where (x,y) denote the distance along and perpendicular to the axis and (u,v) the velocity components in these directions, is

$$(\rho u)_x + (1/y^\epsilon)(y^\epsilon \rho v)_y = 0 \tag{1}$$

where $\epsilon=0$ for two-dimensional flows and $\epsilon=1$ for axial symmetry.

Introducing a stream function ψ defined by

$$\rho u y^\epsilon = \frac{\partial \psi}{\partial y}, \qquad \rho v y^\epsilon = -\frac{\partial \psi}{\partial x} \tag{2}$$

the continuity equation (1) is automatically satisfied. Assuming isoenergetic flow, hence

$$\frac{1}{2}q^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = C \tag{3}$$

and the equation of state is

$$p = \rho^\gamma e^{S/C_v}/\gamma M_\infty^2 \tag{4}$$

The vorticity is given by Crocco's relation,

$$\omega = y^\epsilon P \frac{\mathrm{d}S/R}{\mathrm{d}\psi} \tag{5}$$

Hence, $\omega/y^\epsilon P$ is constant along a streamline.

Combining Eqs. (2) and (5) gives the required stream function equation,

$$\left(\frac{\psi_x}{\rho y^\epsilon}\right)_x + \left(\frac{\psi_y}{\rho y^\epsilon}\right)_y = -y^\epsilon P \frac{\mathrm{d}S/R}{\mathrm{d}\psi} \tag{6}$$

Note that, while the integral formulation of the conservation of mass is based on the Gauss theorem, the integral formulation of the vorticity or stream function equation is based on Stokes' theorem,

$$\Gamma = \oint \boldsymbol{q}\cdot \mathrm{d}\boldsymbol{s} = \int\int \boldsymbol{n}\cdot\omega \mathrm{d}A \tag{7}$$

where $\omega = \nabla \times \boldsymbol{q}$.

Lin and Rubinov[4] show that there is a variational principle for isoenergetic flows, similar to the Bateman principle for irrotational flows, namely that

$$I = \int\int \left[C\rho - \frac{e^{S/C_v}\rho^\gamma}{(\gamma-1)(\gamma M_\infty^2)}\right] y^\epsilon + \frac{1}{2\rho y^\epsilon}(\psi_x^2 + \psi_y^2)\,\mathrm{d}x\mathrm{d}y \tag{8}$$

is stationary with a preassigned function $S/C_v(\psi)$ and a fixed constant C and provided $\rho(x,y)$ and $\psi(x,y)$ are allowed to vary independently.

The condition $\delta I = 0$ leads to

$$\int\int\left\{\left[\left(C - \frac{e^{S/C_v}\rho^{\gamma-1}}{M_\infty^2(\gamma-1)}\right)y^\epsilon - \frac{1}{2\rho^2 y^\epsilon}(\psi_x^2+\psi_y^2)\right]\delta\rho - \left[\frac{\rho^\gamma}{\gamma-1}\frac{e^{S/C_v}}{(\gamma M_\infty^2)}\frac{\mathrm{d}S/C_v}{\mathrm{d}\psi}y^\epsilon + \left(\frac{\psi_x}{\rho y^\epsilon}\right)_x + \left(\frac{\psi_y}{\rho y^\epsilon}\right)_y\right]\delta\psi\right\}\mathrm{d}x\mathrm{d}y + \int \frac{1}{y^\epsilon}\frac{1}{\rho}\frac{\partial\psi}{\partial n}\delta\psi \mathrm{d}\ell = 0 \tag{9}$$

Presented as Paper 81-1017 at the AIAA Fifth Computational Fluid Dynamics Conference, Palo Alto, Calif., June 23-25, 1981; submitted June 29, 1981; revision received June 3, 1982.

*Associate Research Professor. Member AIAA.
†Engineer. Member AIAA.

If proper boundary conditions are satisfied, $\delta I=0$ leads to Bernoulli's equation (3) and the stream function equation (6). For subsonic flows, I has a local minimum provided $S''+S'^2/C_v \leq 0$.[5]

It may be noticed that the integrand in Eq. (8) with ($\epsilon=0$) is equal to $(p+\rho q^2)$, exactly as in the isentropic case. The integral as formulated above can be extended only over a finite region. For an infinite region, a constant or a function independent of $\rho(x,y)$ and $\psi(x,y)$ has to be subtracted from the integrand to make the integral convergent.

To complete the formulation, the entropy increase across the shock is calculated based on

$$\frac{S-S_0}{R}=\frac{1}{\gamma-1}\left\{\log\left(\frac{2\gamma}{\gamma+1}M^2\sin^2\theta-\frac{\gamma-1}{\gamma+1}\right)-\gamma\log\left[\frac{(\gamma+1)M^2\sin^2\theta}{(\gamma-1)M^2\sin^2\theta+2}\right]\right\} \tag{10}$$

where M is the Mach number of the flow upstream of the shock and θ the shock inclination relative to it.

A remark on the weak solution and the conservation form should be made. The Rankine-Hugoniot jump conditions across a shock is

$$[\rho u]\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_s-[\rho v]=0 \tag{11}$$

$$[p+\rho u^2]\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_s-[\rho uv]=0 \tag{12}$$

$$[\rho uv]\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_s-[p+\rho v^2]=0 \tag{13}$$

Equation (11) states that mass flux normal to the shock is conserved, i.e.,

$$[\rho q_n]=0 \tag{14}$$

This condition is satisfied automatically if ψ is continuous across the shock,

$$[\psi]=0 \tag{15}$$

Equations (12) and (13) imply that the normal momentum and the tangential velocity are conserved across the shock,

$$[q_t]=0 \tag{16}$$

$$[\rho q_n^2+p]=0 \tag{17}$$

The tangential velocity condition is satisfied if the stream function equation is solved in conservation form using conservative differencing since its weak solution admits a discontinuity such that

$$\left[\frac{\psi_x}{\rho}\right]\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_s-\left[\frac{\psi_y}{\rho}\right]=0 \tag{18}$$

The conservation of normal momentum across the shock has to be imposed and implemented here through the entropy increase across the shock. A variational principle for a steady flow with finite shocks is discussed in Ref. 6.

Irrotational Flows

Assuming isentropic flow, the vorticity vanishes and the stream function equation reduces for two-dimensional cases to

$$\left(\frac{\psi_x}{\rho}\right)_x+\left(\frac{\psi_y}{\rho}\right)_y=0 \tag{19}$$

or

$$(a^2-u^2)\psi_{xx}-2uv\psi_{xy}+(a^2-v^2)\psi_{yy}=0 \tag{20}$$

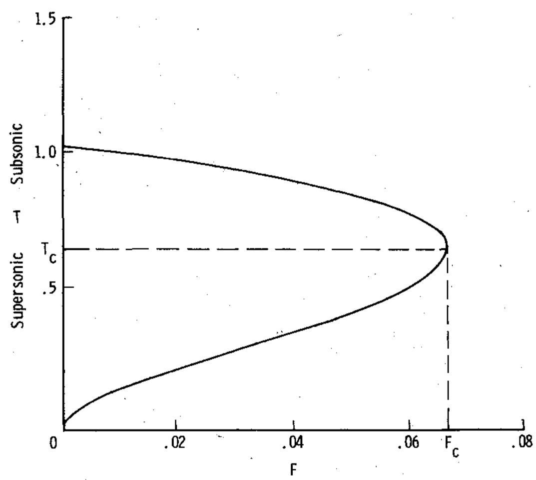
Equation (20) can be rewritten in the symbolic form as

$$(1-M^2)\psi_{ss}+\psi_{nn}=0 \tag{21}$$

where s is the stream direction and n normal to it. The density/mass flow relation is

$$\rho=(M_\infty^2 a^2)^{1/(\gamma-1)}=\left[1-\frac{\gamma-1}{2}M_\infty^2\left(\frac{\psi_x^2+\psi_y^2}{\rho^2}-1\right)\right]^{1/(\gamma-1)} \tag{22}$$

Equation (22) indicates that there are two values of the density for a certain mass flux less than the maximum attainable value. Sells[7] obtained a rational approximation for the subsonic and supersonic branches (see Fig. 1); however, he calculated only subcritical flows.



$\rho_s^{\gamma-1}=1+\frac{1}{2}M^2(\gamma-1),\quad \alpha_s^2=\rho_s^{\gamma-1}/M^2$

$$\frac{f^2}{\rho_s^2\alpha_s^2}=\frac{2}{\gamma-1}\left[1-\left(\frac{\rho}{\rho^2}\right)^{\gamma-1}\right]\left(\frac{\rho}{\rho^2}\right)^2$$

$\rho/\rho_s=\tau,\quad \frac{1}{2}(\gamma-1)f^2/\rho_s^2\alpha_s^2=F$

$$F=\tau^2(1-\tau^{\gamma-1}),\quad F_c=\left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)}\left(\frac{\gamma-1}{\gamma+1}\right)$$

$$\tau_c=\left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)},\quad Q_2=(F_c-F)\frac{2}{(\gamma^2-1)\tau_c^{\gamma+1}}$$

where

$\gamma=1.4,\quad Q=1.44[3(F_c-F)]^{1/2}$

Subsonic

$$\tau=\frac{0.6339395+1.421451Q+0.5431537Q^2-0.1998823Q^3}{1+1.242281Q-0.1511548Q^2-0.0553184Q^3}$$

Supersonic

$$\tau=\frac{0.6339388-0.8257193Q-0.8718085Q^2+1.031123Q^3}{1-0.3022746Q-1.449373Q^2+0.3119677Q^3}$$

Fig. 1 Density mass flux relation after Sells.[7]

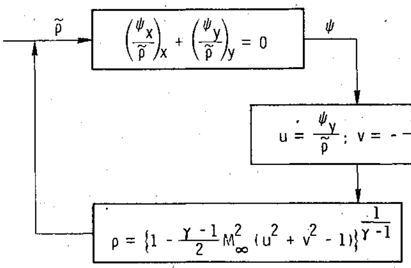


Fig. 2 Iterative algorithm for subsonic flows.

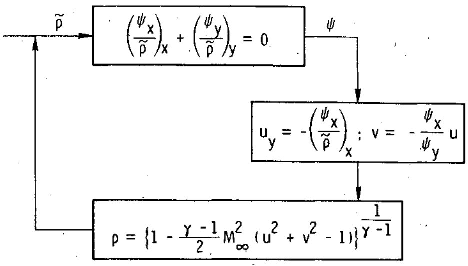


Fig. 3 Iterative algorithm for mixed flows.

Artificial Viscosity Method

Similar to the potential equation, and assuming the Mach number is known, Eq. (20) is solved by adding an artificial viscosity term either explicitly or implicitly through upwind differences in the terms contributing to ψ_{ss}. In the potential calculations, if nonconservative differences are used, mass balance across the shock is not achieved and the error can be represented as a source distribution at the shock. On the other hand, the mass is automatically conserved in the stream function formulation and, due to nonconservative differencing, a vorticity distribution at the shock is produced. The small-disturbance version of Eq. (20) is solved by Chin and Rizetta[8] using the Murman-Cole type dependent differencing and a procedure similar to that of Emmons to calculate the velocity or the Mach number in the supersonic zone. An obvious way to make this calculation conservative is to fit the shock as Emmons did for the full equation since it is not clear how to put the small-disturbance stream function equation in conservation form.

Artificial Density Method

Here, the stream function equation in conservation form is solved using conservative differencing. An artificial viscosity is added through modifying the density,[9]

$$\tilde{\rho} = \rho - \mu \rho_s \Delta s$$

where

$$\rho_s \Delta s \simeq (u/q)\rho_x \Delta x + (v/q)\rho_y \Delta_y$$

and

$$\mu = \max[0, 1 - (1/M^2)] \tag{23}$$

Centered differences are used everywhere, except that ρ_s is evaluated using upwind differences. The stream function equation becomes

$$(\psi_x/\tilde{\rho})_x + (\psi_y/\tilde{\rho})_y = 0 \tag{24}$$

Another version using different artificial densities in the x and y terms is given in Ref. 10. It should be mentioned that the approximation of $\rho_s \Delta s$ is the same as in potential calculations[9] and improved schemes may be obtained by a better approximation.

Iterative Algorithms

In conservative potential calculations, given the density from previous iterations, the potential is calculated from the continuity equation. The density is then updated using Bernoulli's equation. A similar iteration loop is implemented for stream function calculations as shown in Fig. 2. The algorithm converges reliably only for subcritical flows.‡

‡Convergence of transonic flow calculations can sometimes be achieved using the constant μ or a modified version of Eq. (23), namely ($M_c^2 < 1$) $\mu = \max[0, 1 - (M_c^2/M^2)]$. It may be that, due to this form of the artificial viscosity, the numerical model does not possess a singular (critical) point.

The iteration algorithm described in Fig. 2 is based on the relation

$$\rho_{\text{new}} = \left[1 - \frac{\gamma - 1}{2} M_\infty^2 \left(\frac{\psi_x^2 + \psi_y^2}{\rho_{\text{old}}^2} - 1\right)\right]^{1/(\gamma-1)} \tag{25}$$

The density/mass flow relation can be written in a general form

$$F(\rho) = \rho^{\gamma+1} - \left(1 + \frac{\gamma-1}{2} M_\infty^2\right)\rho^2 + \frac{\gamma-1}{2} M_\infty^2 (\psi_x^2 + \psi_y^2) = 0 \tag{26}$$

Application of the Newton method to Eq. (26) gives

$$J(\rho_{\text{old}})(\rho_{\text{new}} - \rho_{\text{old}}) = -F(\rho_{\text{old}})$$

where

$$J(\rho) = (\gamma+1)\rho^\gamma - 2\rho\left(1 + \frac{\gamma-1}{2} M_\infty^2\right) \tag{27}$$

The Jacobian J is negative or positive depending on whether ρ is larger or smaller than ρ^* (ρ at $a = a^*$). For pure subsonic or supersonic flows, Eq. (27) may be used. But for mixed flows, perturbing sonic conditions yield two possible solutions and it is not obvious which one to choose.

Mixed Flow Calculations

Steger[11] used the machine interactively to continue the solution through the sonic line. To avoid the ambiguity in the density updating, a different approach is suggested in Ref. 10, namely to calculate the velocity from the stream function as follows. Initially, the vorticity equation is written in terms of the velocity components

$$u_y - v_x = 0 \tag{28}$$

or

$$(q\cos\theta)_y - (q\sin\theta)_x = 0 \tag{29}$$

where

$$\tan\theta = v/u = -\psi_x/\psi_y$$

Knowing q on a curve crossing the characteristics of Eq. (29), q can be calculated throughout the flowfield step by step (in general, implicit schemes should be used). The conditions along such a curve could be either pure subsonic or pure supersonic, where Newton's iteration may be used. Once q is known, the density is updated through Bernoulli's equation. Such a procedure is needed only in the neighborhood of the sonic line in order to guarantee an analytical continuation of the speed from a subsonic to a supersonic regime consistent with the governing differential equation.

Present Method

The vorticity or stream function equation is rewritten in the form

$$u_y = -(\psi_x/\tilde{\rho})_x \tag{30}$$

Equation (30) is discretized using centered differences everywhere and u is updated in the field using the most recent values of ψ and $\tilde{\rho}$. v is then updated from

$$v = -(\psi_x/\psi_y)u \tag{31}$$

The present method of updating u is consistent with the main calculation of ψ. It is easy to implement, but because of its explicit nature, stability depends on the grid. Again, Eq. (30) needs to be integrated only across the sonic line. Once u and v are known, the density is updated through Bernoulli's equation as before. The iterative algorithm is described in Fig. 3.

Rotational Flows

Method I

One of the main advantages of the stream function formulation is the capability of handling rotational flows. In this study, the vorticity is generated by a curved shock and it can be calculated in terms of the entropy across the shock [Eq. (10)]. Knowing the entropy and the stream function values at the shock locus, a one-dimensional curve fitting procedure is used to establish the entropy stream function relation. At any point downstream of the shock, the entropy can be then calculated in terms of the local value of the stream function from the above relation (since the entropy is constant along a streamline). Not only the entropy $S(\psi)$ is needed but also $\mathrm{d}S(\psi)/\mathrm{d}\psi$.

The entropy is used in the density calculation

$$\rho = \rho_i e^{-(S-S_0)/R} \tag{32}$$

where

$$\rho_i = \left[1 - \frac{\gamma-1}{2} M_\infty^2 (u^2+v^2-1)\right]^{1/(\gamma-1)}$$

The velocity is obtained in a similar way as in the irrotational calculations, namely,

$$u_y = -(\psi_x/\rho)_x - \omega \tag{33}$$

$$v = -(\psi_x/\psi_y)u \tag{34}$$

The vorticity ω is calculated from Eq. (5),

$$\omega = \frac{\rho^\gamma e^{(S-S_0)/C_v}}{\gamma M_\infty^2} \frac{\mathrm{d}(S/R)}{\mathrm{d}\psi} \tag{35}$$

The calculations are made in the following order:

1) Assuming ρ and ω are known, the stream function ψ is updated based on the stream function equation (6).

2) Using the updated value of ψ in Eqs. (33) and (34), u and v are calculated and hence ρ_i.

3) From the new ψ, $S(\psi)$ and $(\mathrm{d}S/R)/\mathrm{d}\psi$ are frequently updated (every 10 or 20 iterations).

4) From steps 2 and 3, ρ is updated. ω is less frequently updated, (every 10 or 20 iterations) and the jump of vorticity across the shock is spread between a few points to obtain reliable convergence.

5) Repeat the above steps until convergence.

In the above method, shocks must be detected to calculate the entropy. A shock detection is simpler than a complete shock-fitting procedure but it is still desirable to avoid such complications.

Method II

Another alternative, where the shock location and inclination are not needed in the course of calculations, is the following. The density is updated from the x-momentum

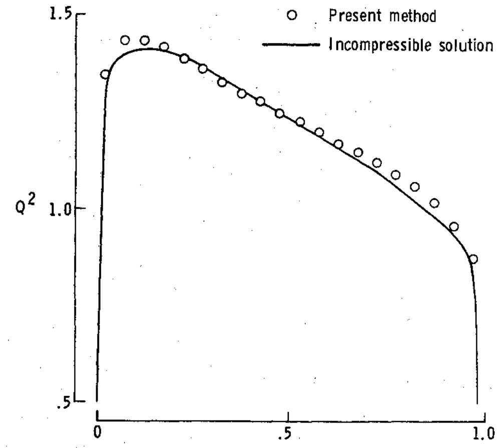


Fig. 4a Surface pressure distribution on NACA 0012 at $M_\infty = 0.1$.

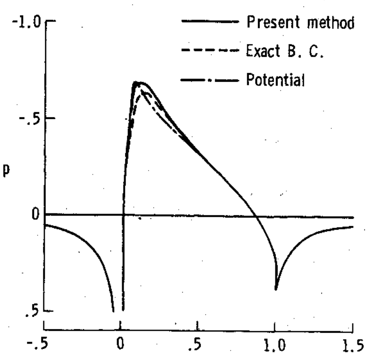


Fig. 4b Surface pressure distribution on NACA 0012 at $M_\infty = 0.7$.

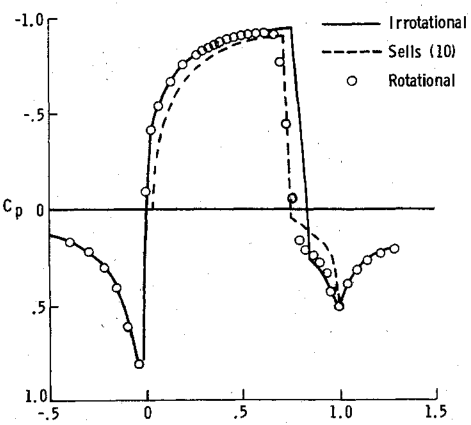


Fig. 4c Surface pressure distribution on NACA 0012 at $M_\infty = 0.85$.

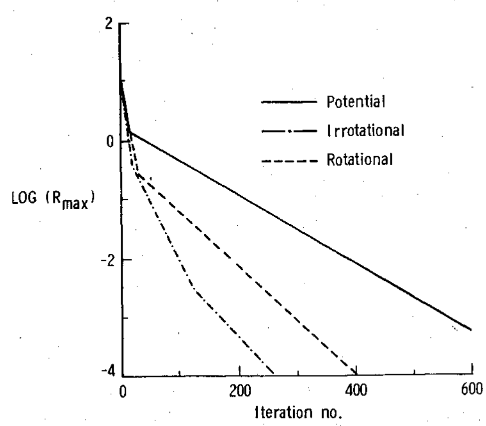


Fig. 4d Rates of convergence of potential, irrotational and rotational stream function calculations (zebroid) of NACA 0012 at $M_\infty = 0.85$.

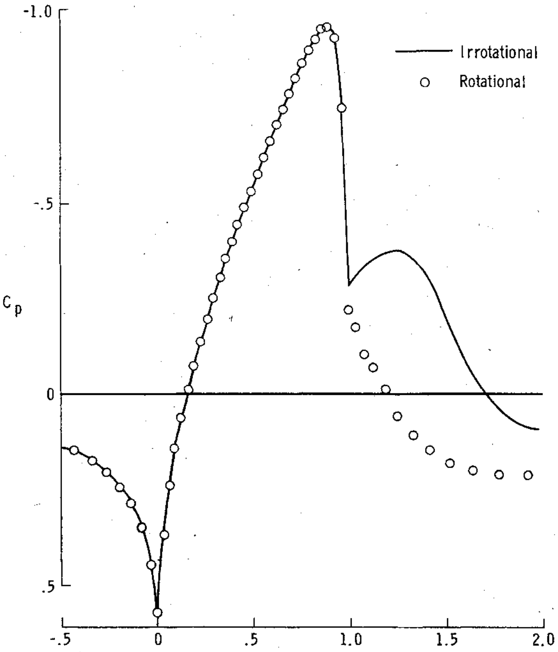


Fig. 5a **Surface pressure distribution on 10% parabolic arc airfoil at** $M_\infty = 0.92$.

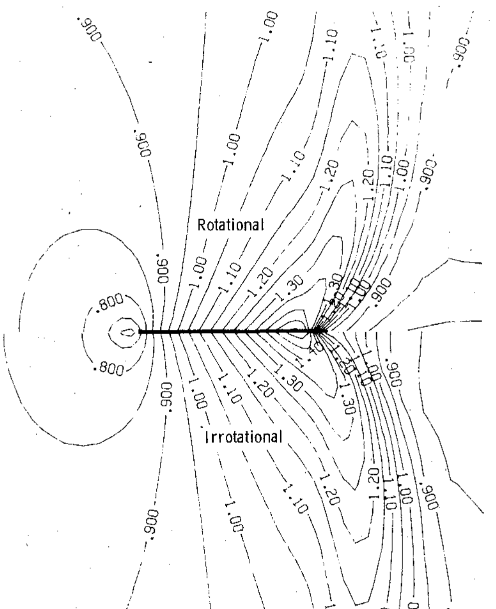


Fig. 5b Mach number contours around 10% parabolic arc airfoil at $M_\infty = 0.92$.

equation (for general body-fitted coordinates, the tangential momentum equation is used),

$$(\rho u^2 + p)_x + (\rho uv)_y = 0 \tag{36}$$

where

$$P = [(\gamma - 1)/\gamma][\rho C - \tfrac{1}{2}\rho q^2]$$

Assuming u and v are known, Eq. (36) is solved for ρ. The entropy is calculated everywhere from

$$(S - S_0)/R = \log(\rho_i/\rho) \tag{37}$$

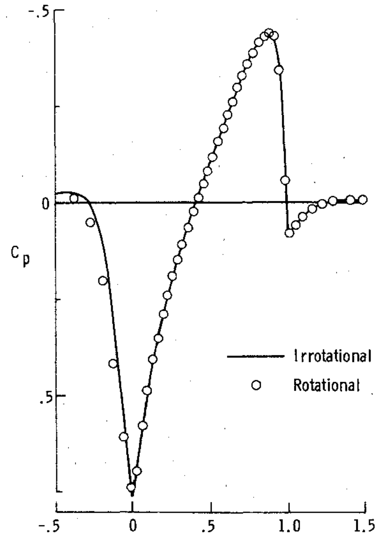


Fig. 5c Surface pressure distribution on 10% parabolic arc airfoil at $M_\infty = 1.2$.

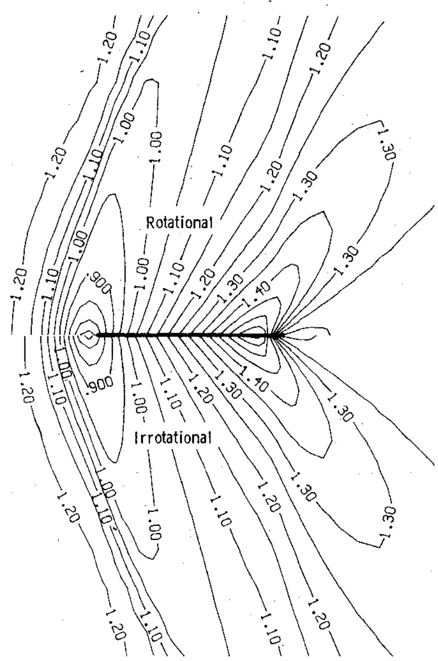


Fig. 5d Mach number contours around 10% parabolic arc airfoil at $M_\infty = 1.2$.

And the vorticity is updated in terms of the density and the entropy where

$$\frac{ds}{d\psi} = \frac{1}{\rho q}\frac{dS}{dn} = \frac{1}{\rho q}\left(\frac{u}{q}\frac{\partial S}{\partial y} - \frac{v}{q}\frac{\partial S}{\partial x}\right) \tag{38}$$

The calculations are made in the following order:

1) Assuming ρ and ω are known, the stream function is updated based on the stream function equation (6).

2) Using the updated value of ψ, u and v are calculated:

$$u = \psi_y/\rho; \quad v = -\psi_x/\rho$$

3) ρ is updated according to the x-momentum equation (36) (using a leapfrog scheme similar to point Zebra[12] and central differencing everywhere for ρ at the cell centers). ρ is slightly under-relaxed to obtain convergence. Other implicit schemes may be used to guarantee unconditional stability for a general grid.

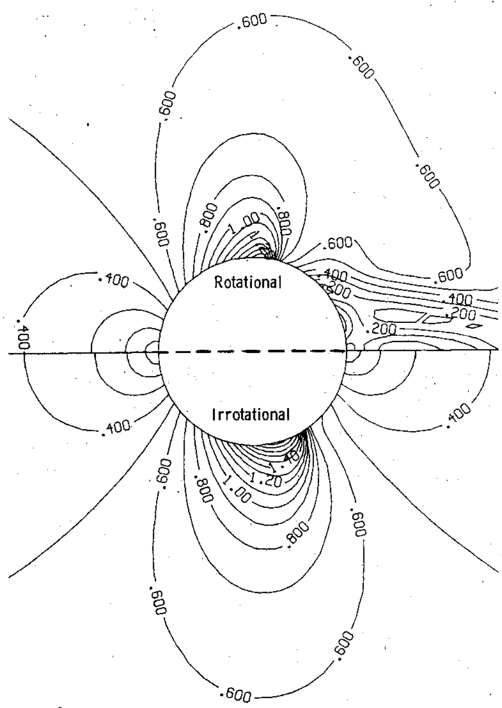


Fig. 6 Mach number contours around a cylinder at $M_\infty = 0.51$.

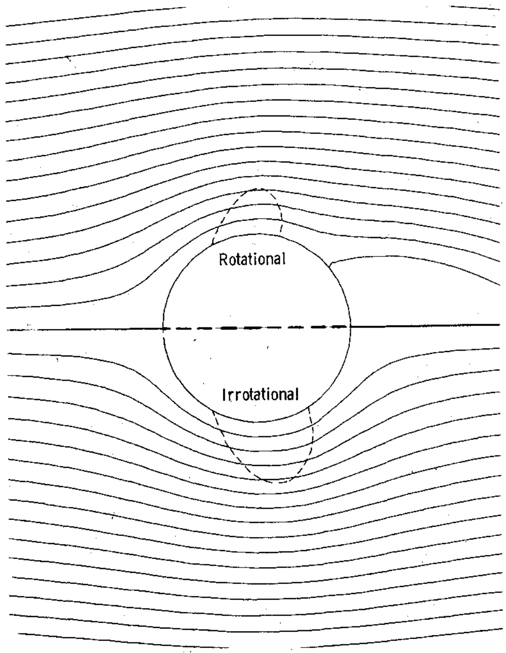


Fig. 7 Streamlines around a cylinder at $M_\infty = 0.51$.

4) Entropy and vorticity are frequently updated using Eqs. (37) and (38).

5) Repeat the above steps until convergence.

In this method, the solution has nonisentropic conditions across a shock even when the vorticity is set equal to zero.

Numerical Results

Transonic flows around an airfoil and a cylinder are calculated based on stream function formulation. Figure 4 shows the pressure distributions around an NACA 0012 airfoil at different Mach numbers. The pressure distributions of the potential and the stream function calculations are slightly different at the leading edge. For $M_\infty = 0.85$ both irrotational and rotational flows are calculated and compared to results obtained from Euler calculations.[12] Rates of convergence of potential, irrotational stream function and rotational stream function calculations are plotted in Fig. 4d. As expected, the stream function calculations are faster because of the Dirichlet boundary condition (the airfoil is a streamline). Transonic flows around a 10% parabolic arc airfoil are presented in Fig. 5. In all airfoil calculations, a linearized boundary condition $[\psi = -f(x)]$ is implemented at $y = 0$. A 61×31 grid is used with a 15% stretching in both the x and y directions. Alternating black and white horizontal lines (zebroid) are solved simultaneously.[12] A ψ_{xt} term is added explicitly to guarantee convergence in the supersonic zone. The rotational calculations reported here are based on method I. Method II has been tested with the NACA 0012 airfoil at $M_\infty = 0.85$ and similar results were obtained, except that the shock was slightly smeared.

Transonic Flows around a Cylinder

Here the exact boundary condition $\psi = 0$ on the surface of the cylinder is used. A 61×31 grid is used with a uniform mesh in the θ direction and 15% stretching in the r direction. The outer boundary condition is placed at a distance approximately 15 times the radius of the cylinder where the flow is assumed parallel. Both irrotational and rotational stream function calculations are presented in Figs. 6 and 7. The same results are obtained for a finer grid (86×43). Figure 8 shows the pressure distribution for different Mach numbers and Fig. 9 the rates of convergence for different calculations. Ac-

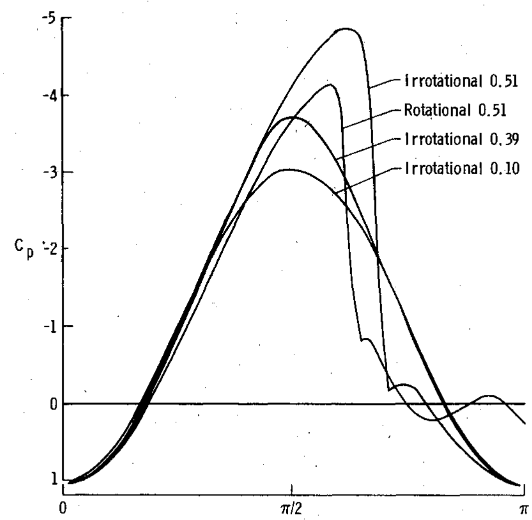


Fig. 8 Pressure distribution around a cylinder at different Mach numbers.

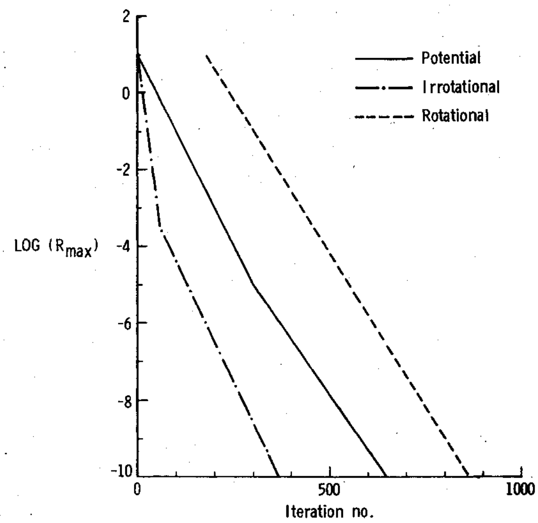


Fig. 9 Rates of convergence of potential, irrotational and rotational stream function calculations (zebroid) of a cylinder at $M_\infty = 0.51$.

celeration of convergence may be achieved with the same methods as used for the potential calculations.

In order to obtain the high Mach number results with closed streamlines, we assume that ω/p is constant in the closed region. Moreover both S and ω/p are assumed continuous across the separating streamline, i.e., for negative ψ; $S/R(\psi) = S/R(0) + C\psi$ where $C = \omega/p(0)$. The first assumption is discussed in the next section. As far as the second assumption is concerned, upon differentiating Eq. (10) with respect to ψ, it is clear that C is finite if $\cos\theta(d\theta/d\psi)$ is. Hence, even if the shock is allowed to have infinite curvature (as, for example, in the work of Zierep[13] or Oswatitsch and Zierep[14]) the vorticity is always assumed finite. In Refs. 13 and 14 the effect of vorticity on the solution at the foot of the shock was neglected. Such analysis is useful for potential flow calculation assessment.[15] Similar development for Euler calculations is under investigation.[16]

Remarks on Inviscid Separation and Closed Streamlines

For steady inviscid, incompressible, rotational flows past an acute corner, local solutions are found provided that the vorticity is known everywhere. Under certain circumstances, the existence of a corner eddy may be inferred. For example, Tsien[17] studied Joukowsky airfoils in shear flows, Fraenkel[18] studied a shear flow past a semicircular projection, and Küchemann[19] studied shear flows past airfoil trailing edges. In all these cases the vorticity is constant everywhere, exact solutions are possible, and conditions for separation are obtained. Recently, Varley and Blythe[20] studied hydraulic shear flows contained between lateral boundaries where the variation of vorticity in the reversed flow region is determined by some consistency conditions. If at some instant the vorticity distribution is specified arbitrarily at all streamlines, generally the subsequent flow will be unsteady.

On the other hand, Prandtl[21] considered laminar flows at high Reynolds number and proved that, within a region of closed streamlines, the vortex assumes a constant value. Batchelor[22] argues that if the viscosity ν is small, convection of vorticity will dominate the viscous diffusion of vorticity when both processes occur, so that ω is approximately constant along the streamlines. But, in exactly steady motion, the net viscous diffusion of vorticity across a closed streamline must be exactly zero and this is then possible only if ω is also approximately constant across the streamlines. It seems that two-dimensional flows with closed streamlines cannot be exactly steady until the slow but persistent effect of the viscous diffusion of vorticity across the streamlines has evened out any variation of vorticity that may have been present initially; the time required for this asymptotic steady state to be set up will, of course, increase as ν decreases. The steady velocity distribution in the closed region cannot be determined by considering the fluid as inviscid. It can be shown that the value of the uniform vorticity in closed flows is determined in one simple case by the condition that the viscous boundary layer surrounding this region must also be in steady motion. More precisely, the governing equation of a steady laminar incompressible flow is

$$q \times \omega - \nabla\left(\frac{p}{\rho} + \frac{1}{2}q^2\right) + \nu\nabla^2 q = \frac{\partial q}{\partial t} = 0 \tag{39}$$

Taking a line integral around a closed streamline, the first two terms vanish and the integral condition is

$$\oint (\nabla \times \omega)\, dS = 0 \tag{40}$$

If viscous forces are neglected, Eq. (39) reduces to

$$q \times \omega = \nabla H \tag{41}$$

where

$$H = p/\rho + \tfrac{1}{2}q^2$$

or

$$\omega = \frac{dH}{d\psi} \tag{42}$$

Substituting Eq. (42) in Eq. (40) gives

$$\frac{d\omega}{d\psi}\oint q\, ds = 0 \tag{43}$$

Hence ω is constant in the closed region.

For the example of a transonic flow around a cylinder, downstream of the shock where the flow is subsonic, Eqs. (34) and hence (40) are approximately valid. Instead of Eq. (42), the vorticity is related to the entropy according to Eq. (5),

$$\omega = p\frac{dS/R}{d\psi} \tag{5}$$

Assuming p is approximately constant in a closed region Eq. (43) becomes

$$\frac{d\omega/p}{d\psi}\oint p\rho q\, ds \simeq 0 \tag{44}$$

Hence, ω/p is approximately constant in a closed region. The value of this constant is underdetermined and there is no further information to be gained from considerations of the region in which viscous forces are small.

Guderley[23] shows that cumulative effects of viscosity and heat conductivity play a role in the final steady flow, even though the changes caused by them during a limited time are very small. The requirement that the cumulative effects of viscosity and heat conduction must vanish enters his analysis as integrability conditions necessary for the existence of a second approximation in a development of the flowfield with respect to $1/Re$ and are re-expressed as the balance equations for momentum, energy, and entropy which are to be satisfied on all $\psi = \text{const}$ lines.

Therefore, the authors believe that solutions obtained by the Euler codes are likely to be dependent on the artificial viscosity used, not only because of the stability of the numerical calculations, but also because of the nonuniqueness problem. Intuitively, the artificial viscosity is responsible for transforming the information across the separating streamline; it is not clear, however, how a purely inviscid solution of the steady Euler equations is uniquely determined. Hopefully, solutions obtained by the Euler codes are the right limit of the Navier-Stokes solutions when the viscosity vanishes.

Remarks on Three-Dimensional Flows

Let ψ and θ be two stream functions[24,25] such that

$$\rho q = \nabla\psi \times \nabla\theta \tag{45}$$

The continuity equation ($\nabla \cdot \rho q = 0$) is automatically satisfied. If the equation of the body is

$$B(x,y,z) = 0 \tag{46}$$

the boundary condition is

$$\nabla\psi \times \nabla\theta \times \nabla B = 0 \tag{47}$$

Hence, the body is a stream surface. Across a shock,

$$[\psi] = [\theta] = 0 \tag{48}$$

and

$$-1:\left(\frac{dx}{dy}\right)_s:\left(\frac{dx}{dz}\right)_s=[u]:[v]:[w]$$

where

$$u=(\psi_y\cdot\theta_z-\theta_y\cdot\psi_z)/\rho$$

$$v=(-\psi_x\cdot\theta_z-\theta_x\cdot\psi_z)/\rho$$

$$w=(\psi_x\cdot\theta_y-\theta_x\cdot\psi_y)/\rho \tag{49}$$

The governing differential equations are

$$w_y-v_z=\omega_1 \qquad u_z-w_x=\omega_2 \qquad v_x-u_y=\omega_3 \tag{50}$$

or, in terms of ψ and θ

$$\left(\frac{\theta_y}{\rho}\cdot\psi_x\right)_y+\left(\frac{\theta_z}{\rho}\cdot\psi_x\right)_z=\left(\frac{\psi_y}{\rho}\cdot\theta_x\right)_y+\left(\frac{\psi_z}{\rho}\cdot\theta_x\right)_z+\omega_1$$

$$\left(\frac{\theta_z}{\rho}\cdot\psi_y\right)_z+\left(\frac{\theta_x}{\rho}\cdot\psi_y\right)_x=\left(\frac{\psi_z}{\rho}\cdot\theta_y\right)_z+\left(\frac{\psi_x}{\rho}\cdot\theta_y\right)_x+\omega_2$$

$$\left(\frac{\theta_x}{\rho}\cdot\psi_z\right)_x+\left(\frac{\theta_y}{\rho}\cdot\psi_z\right)_y=\left(\frac{\psi_x}{\rho}\cdot\theta_z\right)_x+\left(\frac{\psi_y}{\rho}\cdot\theta_z\right)_y+\omega_3 \tag{51}$$

where $\omega=\nabla\times q$ and hence $\nabla\times\omega=0$ or

$$(\omega_1)_x+(\omega_2)_y+(\omega_3)_z=0 \tag{52}$$

The above system reduces to a two-dimensional case when $\theta=z$ and $\psi=\psi(x,y)$.

In general, Bernoulli's equation is

$$h+\tfrac{1}{2}q^2=H$$

where

$$h=e+p/\rho=c_vT+p/\rho=\frac{\gamma}{\gamma-1}p/\rho=C_PT \tag{53}$$

and H is constant along a streamline.

The equation of state is Eq. (4).

Similarly, integral and variational formulations of three-dimensional problems are straightforward. Variational principle for three-dimensional compressible flows is given by Guderley.[26] It is shown that

$$I\iiint(p+\rho q^2)\,dv \text{ is stationary} \tag{54}$$

with the constraints that S and H are constants along streamlines,

$$S=S(\psi,\theta) \tag{55}$$

$$H=H(\psi,\theta) \tag{56}$$

Only for subsonic flow, I has an extremum. The proof is given if the exit points of the streamlines are prescribed and if the downstream extension of the flowfield is limited.

Crocco's relation relates ω to H and S,

$$q\times\omega=\nabla H-T\nabla S \tag{57}$$

or

$$\omega\nabla\psi=\left(\frac{\partial H}{\partial\theta}-T\frac{\partial S}{\partial\theta}\right)\rho \tag{58}$$

$$\omega\nabla\theta=-\left(\frac{\partial H}{\partial\psi}-T\frac{\partial S}{\partial\psi}\right)\rho \tag{59}$$

and in general, $\omega=\omega_s+\omega_H+\omega_L$.

If S and H are constant throughout the flowfield, the components of the vorticity normal to the streamlines vanish, such a vorticity field therefore has the form

$$\omega_L=F\rho q$$

where

$$F=F(\psi,\theta) \tag{60}$$

ω_L refers to the vorticity present in the incoming flow.

Conclusions

The transonic stream function equation in conservation form, taking into account the effects of vorticity generated by the shock, is solved iteratively using the artificial compressibility method. The density is not a unique function of the mass flux. To avoid ambiguity near the sonic line, the density is updated in terms of the velocity, which is obtained through a simple integration of the vorticity equation. The entropy increase across the shock is calculated in terms of the Mach number upstream of the shock and the shock inclination relative to the upstream flow direction. The vorticity along a streamline downstream of the shock is proportional to the pressure and the constant of proportionality is evaluated in terms of the entropy stream function relation.

Another alternative, where the shock needs not to be identified, is to use the tangential-momentum equation (in conservation form) to update the density. The entropy is calculated as the logarithm of the ratio of the isentropic and the nonisentropic densities. The vorticity in this case is evaluated everywhere, its value upstream of the shock is of the order of the truncation error.

In both cases, the numerical solution of Euler equations, is reduced mainly to the solution of a second-order partial differential equation in terms of a stream function, using standard transonic potential methods.

Unlike potential flows, inviscid separation could occur with the stream function formulation. It is assumed that ω/p is constant in the closed region. This constant however, is not determined within the inviscid flow model.

Finally, the problem of three-dimensional flows is formulated in terms of two stream functions.

Acknowledgments

The authors wish to thank Jerry South and Manuel Salas of the Theoretical Aerodynamic Branch of NASA Langley for many helpful discussions. Jerry South provided the authors with his potential relaxation code to start this effort.

References

[1] Emmons, H., "The Numerical Solution of Compressible Fluid Flow Problem," NACA TN 932, 1944.

[2] Emmons, H., "The Theoretical Flow of a Frictionless, Adiabatic, Perfect Gas Inside of a Two-Dimensional Hyperbolic Nozzle," NACA TN 1003, 1946.

[3] Emmons, H., "Flow of a Compressible Fluid Past a Symmetrical Airfoil in a Wind Tunnel and in Free Air," NACA TN 1746, 1948.

[4] Lin, C. C. and Rubinov, L., "On the Flow Behind Curved Shocks," *Journal of Mathematics and Physics*, Vol. 27, 1948, pp. 105-129.

[5] Morice, P., "Un Principe Variationnel et une Methode de Résolution Numérique par Éléments Finis pour des Écoulements Avec Frontieres Libres," Paper presented at XIIIe Symposium of Dynamics of Fluids, Kortowo, Poland, Sept. 1977.

[6] Manwell, A. R., "A Variational Principle for Steady Homenergic Compressible Flow with Finite Shocks," *Wave Motion*, Vol. 2, 1980, pp. 83-95.

[7] Sells, C. C., "Plane Subcritical Flow Past a Lifting Aerofoil," *Proceedings of Royal Society*, Vol. A308, 1968, pp. 377-401.

[8]Chin, W. and Rizetta, D. P., "Airfoil Design in Subcritical and Supercritical Flows," *Journal of Applied Mechanics*, Vol. 46, Sept. 1979, pp. 761-766.

[9]Hafez, M., Murman, E. M., and South, J., "Artificial Compressibility Methods for Numerical Solution of Transonic Full Potential Equation," AIAA Paper 78-1148, 1978.

[10]Hafez, M., "Numerical Solution of Transonic Full Stream Function Equation in Conservation Form," Flow Research Note 178, 1979.

[11]Steger, J. L., "Application of Cyclic Relaxation Procedures to Transonic Flow Fields," Ph.D. Thesis, Iowa State University, Ames, 1969.

[12]Hafez, M. and South, J. C., "Vectorization of Relaxation Methods for Solving Transonic Full Potential Equation," *Numerical Methods for the Computation of Inviscid Transonic Flow with Shock Waves*, GAMM Workshop, Sweden, 1979.

[13]Zierep, J., "Der Senkrechte Verdichtungsstoss am gekrümmten Profil," *Zeitschrift fuer Angewandte Mathematik und Physik*, Vol. 96, 1958, pp. 764-776.

[14]Oswatitsch, K. and Zierep, J., "Das Problem des senkrechten Stosses an einer gekrummten Wand," *ZAMM*, Vol. 40, 1960, p. 143.

[15]Hafez, M. and Cheng, H. K., "Shock Fitting Applied to Relaxation Solutions of Transonic Small-Disturbance Equations," *AIAA Journal*, Vol. 15, June 1977, pp. 786-793.

[16]Fung, K. Y., "On the Vorticity at a Shock Foot in Inviscid Flow," to appear.

[17]Tsien, H. S., "Symmetrical Joukowsky Airfoils in Shear Flows," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. I, 1943, pp. 130-148.

[18]Fraenkel, L. E., "On Corner Eddies in Plane Inviscid Shear Flow," *Journal of Fluid Mechanics*, Vol. II, 1961, pp. 400-406.

[19]Küchemann, D., "Inviscid Shear Flow Near the Trailing Edge of an Airfoil," *Zeitschrift fuer Flugwissenschaften*, Vol. 15, 1967, pp. 292-294.

[20]Varley, E. and Blythe, P. A., "Long Eddies in Sheared Flows," *Journal of Fluid Mechanics*, to appear.

[21]Prandtl, L., "Motion of Fluids with Very Little Viscosity," NACA TM 452, 1928.

[22]Batchelor, G. K., "On Steady Laminar Flow with Closed Streamlines at Large Reynolds Number," *Journal of Fluid Mechanics*, Vol. 1, 1956, pp. 177-190.

[23]Guderley, K. G., "Steady Compressible Swirl Flows with Closed Streamlines at High Reynolds Number," *Journal of Fluid Mechanics*, Vol. 82, 1977, pp. 321-347.

[24]Giese, J. H., "Stream Functions for Three-Dimensional Flows," *Journal of Mathematics and Physics*, Vol. 30, 1951, pp. 31-35.

[25]Skobelkin, V. I., "Variational Principles in Hydrodynamics," *Soviet Physics—JETP*, Vol. 4, No. 1, 1957, pp. 68-73.

[26]Guderley, K. G., "An Extremum Principle for Three Dimensional Compressible Inviscid Flows," *SIAM Journal of Applied Mathematics*, Vol. 23, No. 2, 1972, pp. 259-275.